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Elementary derivations of the Rogers-Fine identity and other q-series identities

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Dedicated to Professor Stephen C. Milne on the occasion of his 75th birthday

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ABSTRACT

We begin the article with a proof of the Rogers-Fine identity. We then show that the Rogers-Fine identity implies the Rogers-Ramanujan identities as well as a new finite version of the quintuple identity. Motivated by the connections between these identities, we discover an identity which yields proofs of Rogers-Ramanujan-type identities associated with the Rogers-Ramanujan continued fraction, the Ramanujan-Göllnitz-Gordon continued fraction and Ramanujan's cubic continued fraction. We also discover a new generalization of the quintuple product identity which leads to a generalization of an identity due to R.J. Evans and a short proof of q-Chu-Vandermonde identity that does not require the knowledge of the q-binomial theorem.

1. Introduction

In this article, we revisit and provide elementary proofs to classical identities such as the Rogers-Fine identity, the Rogers-Ramanujan identities, the quintuple product identity, the Ramanujan-Göllnitz-Gordon identities, the Ramanujan identities associated with his cubic continued fraction, Evans' identity for certain Lambert series, the q-Chu-Vandermonde identity and the Jacobi triple product identity.

In Section 2, we present a new proof of the Rogers-Fine identity.

In Section 3, we use the Rogers-Fine identity to derive an identity that is subsequently used to establish the Rogers-Ramanujan identity and a new finite version of the quintuple identity.

In Section 4, we discuss an extension of the identity established in Section 3 to prove the Ramanujan-Göllnitz-Gordon identities.

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For $|q|<1$ and $a \in \mathbb{C}$, let

$$(a;q)_0=1,\ (a;q)_\infty=\prod_{k=1}^{\infty}(1-aq^{k-1}) \text{ and } (a_1,a_2,\ldots,a_k;q)_\infty=\prod_{i=1}^{k}(a_i;q)_\infty.$$

In addition, for any nonzero integer j, let

$$(a;q)_j=\frac{(a;q)_\infty}{(aq^j;q)_\infty} \text{ and } (a_1,a_2,\ldots,a_k;q)_j=\prod_{i=1}^{k}(a_i;q)_j.$$

In Section 5, we derive the identity

$$\sum_{n=0}^{N}\frac{(a/c;q^2)_n(-c)^nq^{n^2+n}}{(q^2;q^2)_{N-n}(q^2,\ bq^2;q^2)_n}=(cq^2;q^2)_N\sum_{n=0}^{N}\frac{(a,a/b,a/c;q^2)_n(1-aq^{4n})(-bc)^nq^{3n^2+n}}{(q^2;q^2)_{N-n}(a;q^2)_{N+n+1}(q^2,bq^2,cq^2;q^2)_n}$$

using an elementary approach and use it to derive all the identities associated with the Rogers-Ramanujan identity, the Ramanujan-Göllnitz-Gordon identity and Ramanujan's identities associated to his cubic continued fraction. We emphasize that our proofs of these identities, unlike most proofs presented in the literature, do not require the use of Watson's q-analogue of Whipple's theorem, also known as Watson's ${}_8\varphi_7$ transformation formula. (See for example, [3, p. 81, (4.1.3)] and [16, p. 242, (III.17)].)

We also derive a generalization of the finite version of the quintuple identity in Section 3. This generalization implies that

$$\frac{1}{(yq/x,zq;q)_\infty}\sum_{j=-\infty}^{\infty}\frac{(x/z;q)_j(-z)^jq^{(j^2+j)/2}}{(yq;q)_j}=\frac{1}{(zq/x,yq;q)_\infty}\sum_{j=-\infty}^{\infty}\frac{(x/y;q)_j(-y)^jq^{(j^2+j)/2}}{(zq;q)_j}. \tag{1.1}$$

If we set $x=yz$ in (1.1), we deduce that

$$(q/y,y;q)_\infty\sum_{j=-\infty}^{\infty}\frac{(-z)^jq^{(j^2+j)/2}}{1-yq^j}=(q/z,z;q)_\infty\sum_{j=-\infty}^{\infty}\frac{(-y)^jq^{(j^2+j)/2}}{1-zq^j}. \tag{1.2}$$

Identity (1.2) can be found in R.J. Evans' article [13, (1. 8)] and S.H. Chan's article [10, Theorem 4.2]. Evans gave two proofs of (1.2), with the first proof using Ramanujan's ${}_1\psi_1$-summation formula and the second proof using Bailey's ${}_2\psi_2$-summation formula. Chan's proof, on the other hand, involves the partial fraction decomposition of certain quotients of finite products (see [10, (2.1), (4.4)]). We emphasize here (1.1) cannot be established using the methods of Evans' proof or Chan's proof of (1.2).

The q-Chu-Vandermonde identity plays a crucial role in this article. In Section 6, we provide an elementary proof of the q-Chu-Vandermonde identity that does not depend on the use of q-binomial theorem. We then show that the q-Chu-Vandermonde identity implies the Jacobi triple product identity.

2. The Rogers-Fine identity

The Rogers-Fine identity is stated as follows.

Theorem 2.1. *Suppose $|q|<1$, $|t|<1$ and $b \neq q^{-m}$ for any non-negative integer m. Then*

$$(1-t)\sum_{j=0}^{\infty}\frac{(a;q)_j}{(b;q)_j}t^j=\sum_{j=0}^{\infty}\frac{(a,atq/b;q)_j}{(b,tq;q)_j}(bt)^jq^{j^2-j}(1-atq^{2j}). \tag{2.1}$$

There are several proofs of (2.1) (see [2, Section 9.1]) and one of the simplest proofs is given by N.J. Fine [14, p. 15]. We now present a new proof of (2.1) by studying the right-hand side of (2.1).

Proof. For simplicity we use $F(a,b,t)$ to denote the right-hand side of (2.1). Substituting the identity

$$(1-atq^{2j})=(1-tq^j)+tq^j(1-aq^j)$$

into the right-hand side of (2.1) we find that

$$
\begin{aligned}
F(a,b,t) &= \sum_{j=0}^{\infty} \frac{(a,atq/b;q)_j (bt)^j q^{j^2-j}}{(b;q)_j (tq;q)_{j-1}} + t\sum_{j=0}^{\infty} \frac{(a;q)_{j+1}(atq/b;q)_j (bt)^j q^{j^2}}{(b,tq;q)_j} \\
&= 1-t+\sum_{j=1}^{\infty} \frac{(a,atq/b;q)_j}{(tq;q)_{j-1}(b;q)_j}(bt)^j q^{j^2-j} + t\sum_{j=0}^{\infty} \frac{(a;q)_{j+1}(atq/b;q)_j (bt)^j q^{j^2}}{(b,tq;q)_j} \\
&= 1-t+\sum_{j=0}^{\infty} \frac{(a,atq/b;q)_{j+1}}{(b;q)_{j+1}(tq;q)_j}(bt)^{j+1} q^{j^2+j} + t\sum_{j=0}^{\infty} \frac{(a;q)_{j+1}(atq/b;q)_j (bt)^j q^{j^2}}{(b,tq;q)_j} \\
&= 1-t+t\sum_{j=0}^{\infty} \frac{(a;q)_{j+1}(atq/b;q)_j}{(b,tq;q)_j}(bt)^j q^{j^2}\left(1+\frac{bq^j(1-atq^{j+1}/b)}{1-bq^j}\right) \\
&= 1-t+t\frac{(1-a)}{(1-b)}\sum_{j=0}^{\infty} \frac{(aq,atq/b;q)_j}{(bq,tq;q)_j}(1-atq^{2j+1})(bt)^j q^{j^2} \\
&= 1-t+t\frac{(1-a)}{(1-b)}F(aq,bq,t).
\end{aligned} \tag{2.2}
$$

Replacing (a,b) by (aq^n,bq^n) in (2.2), we find that

$$
F(aq^n,bq^n,t)-\frac{(1-aq^n)}{(1-bq^n)}tF(aq^{n+1},bq^{n+1},t)=(1-t). \tag{2.3}
$$

Multiplying both sides of (2.3) by

$$
\frac{(a;q)_n}{(b;q)_n}t^n,
$$

we conclude that

$$
\frac{(a;q)_n}{(b;q)_n}t^n F(aq^n,bq^n,t)-\frac{(a;q)_{n+1}}{(b;q)_{n+1}}t^{n+1}F(aq^{n+1},bq^{n+1},t)=(1-t)\frac{(a;q)_n}{(b;q)_n}t^n. \tag{2.4}
$$

Summing (2.4) from $n=0$ to $n=N$, we deduce that

$$
F(a,b,t)-\frac{(a;q)_{N+1}}{(b;q)_{N+1}}t^{N+1}F(aq^{N+1},bq^{N+1},t)=(1-t)\sum_{n=0}^{N}\frac{(a;q)_n}{(b;q)_n}t^n. \tag{2.5}
$$

Since $|t|<1$ and $F(0,0,t)=1$, we deduce that

$$
\lim_{N\to\infty}\frac{(a;q)_{N+1}}{(b;q)_{N+1}}t^{N+1}F(aq^{N+1},bq^{N+1},t)=\frac{(a;q)_\infty}{(b;q)_\infty}F(0,0,t)\lim_{N\to\infty}t^{N+1}=0.
$$

This shows that letting $N\to\infty$ in (2.5) yields (2.1). □

From (2.1), we deduce the following corollary.

Corollary 1. *Suppose $|q|<1$ and $b\neq q^{-k}$ for non-negative integer k. Then*

$$
\frac{(a;q)_\infty}{(b;q)_\infty}=\sum_{j=0}^{\infty}\frac{(a;q)_j(aq/b;q)_j}{(b;q)_j(q;q)_j}(1-aq^{2j})b^j q^{j^2-j}. \tag{2.6}
$$

Proof. First we rewrite (2.1) as

$$
\sum_{j=0}^{\infty}\frac{(a;q)_j}{(b;q)_j}t^j-\sum_{j=0}^{\infty}\frac{(a;q)_j}{(b;q)_j}t^{j+1}=\sum_{j=0}^{\infty}\frac{(a,atq/b;q)_j}{(b,tq;q)_j}(bt)^j q^{j^2-j}(1-atq^{2j}). \tag{2.7}
$$

Letting $t\to 1$ in the left-hand side of (2.7), we deduce that

$$\begin{aligned}\lim_{t\to1}\left(\sum_{j=0}^{\infty}\frac{(a;q)_j}{(b;q)_j}t^j-\sum_{j=0}^{\infty}\frac{(a;q)_j}{(b;q)_j}t^{j+1}\right)&=1+\lim_{t\to1}\lim_{N\to\infty}\sum_{j=0}^{N}\left(\frac{(a;q)_{j+1}}{(b;q)_{j+1}}-\sum_{j=0}^{\infty}\frac{(a;q)_j}{(b;q)_j}\right)t^{j+1}\\&=1+\lim_{N\to\infty}\lim_{t\to1}\sum_{j=0}^{N}\left(\frac{(a;q)_{j+1}}{(b;q)_{j+1}}-\sum_{j=0}^{\infty}\frac{(a;q)_j}{(b;q)_j}\right)t^{j+1}\\&=1+\lim_{N\to\infty}\sum_{j=0}^{N}\left(\frac{(a;q)_{j+1}}{(b;q)_{j+1}}-\frac{(a;q)_j}{(b;q)_j}\right)=\frac{(a;q)_\infty}{(b;q)_\infty}.\end{aligned}$$

Identity (2.6) follows by letting $t\to1$ in the right-hand side of (2.7). □

Observe that if we let $b\to0$ in (2.6), we obtain the identity

$$(a;q)_\infty=\sum_{j=0}^{\infty}(-1)^j\frac{(a;q)_j}{(q;q)_j}a^jq^{(3j^2-j)/2}(1-aq^{2j}). \tag{2.8}$$

Identity (2.8), which can be found in [14, (13.4)], may be viewed as a generalization of Euler's pentagonal identity

$$\prod_{k=1}^{\infty}(1-q^k)=\sum_{j=0}^{\infty}(-1)^jq^{(3j^2+j)/2}+\sum_{j=1}^{\infty}(-1)^jq^{(3j^2-j)/2}$$

since the latter can be obtained by substituting $a=q$ in (2.8).

3. The Rogers-Fine identity, the Rogers-Ramanujan identities and the quintuple product identity

The well-known Rogers-Ramanujan identities are given by

$$\sum_{j=0}^{\infty}\frac{q^{j^2}}{(q;q)_j}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{3.1}$$

and

$$\sum_{j=0}^{\infty}\frac{q^{j^2+j}}{(q;q)_j}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{3.2}$$

There are now many proofs of (3.1) and (3.2) but we highlight two ways of establishing these identities.

The first approach is due to D.M. Bressoud [6]. In his proofs, the identity [5, p. 77, Proposition 3.4]

$$\frac{1}{(xq;q)_n}=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\frac{x^jq^{j^2}}{(xq;q)_j} \tag{3.3}$$

where

$$\begin{bmatrix}N\\m\end{bmatrix}_q=\frac{(q;q)_N}{(q;q)_m(q;q)_{N-m}}, \tag{3.4}$$

plays an important role. Identity (3.3) is proved by induction in [5] but it is essentially a consequence of

$$\frac{(c/b;q)_N}{(c;q)_N}=\sum_{m=0}^{N}\begin{bmatrix}N\\m\end{bmatrix}_q(-1)^mq^{m(m-1)/2}\frac{(b;q)_m}{(c;q)_m}\left(\frac{c}{b}\right)^m. \tag{3.5}$$

Identity (3.5) is known as the q-analogue of the Chu-Vandermonde identity. The simplest proof of (3.5) using the q-binomial theorem

$$\sum_{j=0}^{\infty}\frac{(a;q)_j}{(q;q)_j}z^j=\frac{(az;q)_\infty}{(z;q)_\infty}$$

is given by Gasper [15], [18, (3.7), (3.12), (3.13)]. For a proof using Heine's transformation formula, which also relies on the q-binomial theorem, see [16, Section 1.5]. In Section 6, we give a new proof of (3.5) without appealing to the q-binomial theorem.

The second approach is by observing that (3.1) and (3.2) are consequences of

$$(aq;q)_\infty \sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(q;q)_j} = \sum_{j=0}^{\infty} \frac{(-1)^j (a;q)_j (1-aq^{2j}) a^{2j} q^{(5j^2-j)/2}}{(q;q)_j (1-a)}. \tag{3.6}$$

Indeed, (3.1) and (3.2) follow from (3.6) by setting $a=1$ and $a=q$, respectively. A proof of (3.6) can be found in [4, p. 77]. According to R.A. Askey, S. Ramanujan could have established (3.1) and (3.2) by using an entry [4, p. 16, Entry 7] in his notebooks. These proofs that Ramanujan missed were discovered by G.N. Watson [21], who proved the identities using his ${}_8\varphi_7$-transformation formula. In a recent article, H.C. Chan revisited (3.6) and provided an "elementary" proof of (3.6) by establishing an identity involving q^{k^2} [7, (1.4)]. In this section, we give another new and elementary proof of (3.6) as an application of the Rogers-Fine identity.

In (2.6), let $b=aq^{n+1}$, where n is a positive integer. We find that

$$(a;q)_{n+1} = \sum_{j=0}^{n} \frac{(a;q)_j (q^{-n};q)_j}{(q,aq^{n+1};q)_j} a^j (1-aq^{2j}) q^{j^2+jn}. \tag{3.7}$$

From (3.7), we deduce that

$$\begin{aligned} 1 &= \sum_{j=0}^{n} \frac{(q^{-n},a;q)_j}{(q;q)_j (a;q)_{n+j+1}} a^j (1-aq^{2j}) q^{j^2+nj} \\ &= \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(a;q)_j a^j}{(a;q)_{n+j+1}} (1-aq^{2j}) q^{(3j^2-j)/2}. \end{aligned} \tag{3.8}$$

Next, we observe that

$$\begin{aligned} &\sum_{n=0}^{N} \frac{a^n q^{n^2}}{(q;q)_n (q;q)_{N-n}} \\ &= \sum_{n=0}^{N} \frac{a^n q^{n^2}}{(q;q)_n (q;q)_{N-n}} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(a;q)_j a^j}{(a;q)_{n+j+1}} (1-aq^{2j}) q^{(3j^2-j)/2} \\ &= \sum_{j=0}^{N} \frac{(-1)^j (a;q)_j a^j (1-aq^{2j}) q^{(3j^2-j)/2}}{(q;q)_j} \sum_{n=j}^{N} \frac{a^n q^{n^2}}{(q;q)_{n-j} (a;q)_{n+j+1} (q;q)_{N-n}} \\ &= \sum_{j=0}^{N} (-1)^j \frac{(1-aq^{2j})(a;q)_j a^{2j}}{(q;q)_j (q;q)_{N-j} (a;q)_{2j+1}} q^{(5j^2-j)/2} \sum_{m=0}^{N-j} \begin{bmatrix} N-j \\ m \end{bmatrix}_q q^{m(m-1)} \frac{(aq^{2j+1})^m}{(aq^{2j+1};q)_m}. \end{aligned} \tag{3.9}$$

where we have used (3.8) in the first equality.

Letting $b\to\infty$ in (3.5) and then replacing N by $N-j$ and finally setting $c=aq^{2j+1}$, we deduce that

$$\sum_{m=0}^{N-j} \begin{bmatrix} N-j \\ m \end{bmatrix}_q q^{m(m-1)} \frac{(aq^{2j+1})^m}{(aq^{2j+1};q)_m} = \frac{1}{(aq^{2j+1};q)_{N-j}}. \tag{3.10}$$

Substituting (3.10) into (3.9), we deduce that

$$\sum_{n=0}^{N} \frac{a^n q^{n^2}}{(q;q)_n (q;q)_{N-n}} = \sum_{j=0}^{N} (-1)^j \frac{(1-aq^{2j})(a;q)_j a^{2j}}{(q;q)_j (q;q)_{N-j} (a;q)_{N+j+1}} q^{\frac{1}{2}(5j^2-j)}. \tag{3.11}$$

Letting $N\to\infty$ in (3.11), we conclude that

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n} = (a;q)_\infty^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{(a;q)_n (1-aq^{2n})}{(q;q)_n} a^{2n} q^{\frac{1}{2}n(5n-1)} \tag{3.12}$$

and this completes the proof of (3.6).

We have shown that (3.1) and (3.2) can be derived from the Rogers-Fine identity (2.1) and a consequence of the q-Chu-Vandermonde identity (3.10). In Section 5, we show how to derive (3.10) without using the q-Chu-Vandermonde identity (see the proof of (5.8)).

We end this section by proving a new finite form of the quintuple product identity using (3.11). Our discovery of the new identity is inspired by the proof (known to C.F. Gauss and A.-L. Cauchy) of the Jacobi triple product identity [1, Theorem 10.4.1]

$$(x;q)_\infty (q/x;q)_\infty (q;q)_\infty = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(j-1)/2} x^j. \tag{3.13}$$

Theorem 3.1. *For any nonnegative integer N and $x \neq q^m$ for $m \in \mathbf{Z}$, we have*

$$\sum_{n=-N}^{N} \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q \frac{(1-xq^{2n})q^{3n^2-n}x^{3n}}{(x;q)_{N+n+1}(q/x;q)_{N-n}} = \sum_{n=-N}^{N} \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q q^{n^2} x^n. \tag{3.14}$$

Proof. Using (3.4), (3.11) can be written as

$$\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_q a^n q^{n^2} = \sum_{n=0}^{N} (-1)^n \begin{bmatrix} N \\ n \end{bmatrix}_q \frac{(1-aq^{2n})(a;q)_n a^{2n}}{(a;q)_{N+n+1}} q^{\frac{1}{2}(5n^2-n)}. \tag{3.15}$$

Replacing N by $2N$ in (3.15) and making the change of variable of $n \to N+n$, we find that

$$\sum_{n=-N}^{N} \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q a^{n+N} q^{(n+N)^2} = \sum_{n=-N}^{N} \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q \frac{(-1)^{n+N} a^{2n+2N}(1-aq^{2n+2N})(a;q)_{n+N} q^{5(n+N)^2/2-(n+N)/2}}{(a;q)_{3N+n+1}}. \tag{3.16}$$

Simplifying (3.16), we deduce that

$$\begin{aligned}\sum_{n=-N}^{N} \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q a^n q^{n^2+2nN} &= \frac{(-a)^N q^{(3N^2-N)/2}(a;q)_N}{(a;q)_{3N+1}} \\ &\times \sum_{n=-N}^{N} \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q \frac{(-1)^n a^{2n}(1-aq^{2n+2N})q^{(5n^2-n)/2+5nN}(aq^N;q)_n}{(aq^{3N+1};q)_n}.\end{aligned} \tag{3.17}$$

Setting $a = xq^{-2N}$ in (3.17), we arrive at

$$\begin{aligned}\sum_{n=-N}^{N} \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q q^{n^2} x^n &= (-x)^N q^{-N(N+1)/2} \frac{(xq^{-2N};q)_N}{(xq^{-2N};q)_{3N+1}} \\ &\times \sum_{n=-N}^{N} (-1)^n \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q \frac{(1-xq^{2n})(xq^{-N};q)_n x^{2n}}{(xq^{N+1};q)_n} q^{(5n^2-n)/2+nN}.\end{aligned} \tag{3.18}$$

Using the definition of the q-shifted factorial, we find that

$$\begin{aligned}(xq^{-2N};q)_{3N+1} &= (xq^{-2N};q)_N (xq^{-N};q)_{2N+1} \\ &= (xq^{-2N};q)_N (xq^{-N};q)_N (x;q)_{N+1} \\ &= (-x)^N q^{-N(N+1)/2} (xq^{-2N};q)_N (q/x;q)_N (x;q)_{N+1}.\end{aligned}$$

It follows that

$$(-x)^N q^{-N(N+1)/2} \frac{(xq^{-2N};q)_N}{(xq^{-2N};q)_{3N+1}} = \frac{1}{(x;q)_{N+1}(q/x;q)_N}. \tag{3.19}$$

Using (3.19), we simplify (3.18) as

$$\sum_{n=-N}^{N} \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q q^{n^2} x^n = \frac{1}{(x;q)_{N+1}(q/x;q)_N} \sum_{n=-N}^{N} (-1)^n \begin{bmatrix} 2N \\ N+n \end{bmatrix}_q \frac{(1-xq^{2n})(xq^{-N};q)_n x^{2n}}{(xq^{N+1};q)_n} q^{(5n^2-n)/2+nN}. \tag{3.20}$$

Since

$$(xq^{-N};q)_n = (-x)^n q^{-nN+n(n-1)/2} (q^{N-n+1}/x;q)_n$$

for any integer n, we observe that (3.20) implies (3.14). □

Letting $N \to \infty$ in (3.14) and then using the Jacobi triple product identity (3.13), we arrive at the quintuple product identity

$$\sum_{n=-\infty}^{\infty}(1-xq^{2n})q^{3n^2-n}x^{3n} = (x, q/x; q)_\infty \sum_{n=-\infty}^{\infty} q^{n^2}x^n$$
$$= (x, q/x; q)_\infty (q^2, -qx, -q/x; q^2)_\infty.$$

Using the definition of the q-shifted factorial we find that

$$(x; q)_\infty = (x, qx; q^2)_\infty \text{ and } (q/x; q)_\infty = (q/x, q^2/x; q^2)_\infty.$$

Thus we have

$$(x, q/x; q)_\infty (q^2, -qx, -q/x; q^2)_\infty = (q^2, x, q^2/x; q^2)_\infty (q^2x^2, q^4/x^2; q^4)_\infty.$$

It follows that

$$\sum_{n=-\infty}^{\infty}(1-xq^{2n})q^{3n^2-n}x^{3n} = (q^2, x, q^2/x; q^2)_\infty (q^2x^2, q^4/x^2; q^4)_\infty. \tag{3.21}$$

Replacing n by $-n$, q^2 by q and x by $1/x$ in (3.21), we arrive at another version of the quintuple product identity [12, (1.11)]

$$\sum_{n=-\infty}^{\infty}(1-x^{-1}q^{-n})q^{(3n^2+n)/2}x^{3n} = (q, 1/x, qx; q)_\infty (qx^2, q/x^2; q^2)_\infty. \tag{3.22}$$

Identity (3.14) is not the only finite version of the quintuple identity (3.22). For other finite versions of (3.22), see [19, (27)], [11] and [17, Sec. 2.10].

Remark 3.1. Our discovery of (3.14) shows that the Rogers-Fine identity, the Rogers-Ramanujan identities and the quintuple product identity are connected via a single identity (3.11).

4. New proofs of the Ramanujan-Göllnitz-Gordon identities

Ramanujan's interest in (3.1) and (3.2) might have originated from his interest in proving his continued fraction

$$\frac{e^{-2\pi/5}}{1} + \frac{e^{-2\pi}}{1} + \frac{e^{-4\pi}}{1} + \frac{e^{-6\pi}}{1} \cdots = \sqrt{\frac{\sqrt{5}+5}{2}} - \frac{\sqrt{5}+1}{2}, \tag{4.1}$$

which is the value of the Rogers-Ramanujan continued fraction

$$F(q) = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} \cdots$$

at $q = e^{-2\pi}$. The identities (3.1) and (3.2) are crucial in proving (4.1).

Ramanujan's continued fraction (4.1) has an interesting "companion" known as the Ramanujan-Göllnitz-Gordon continued fraction given by [9, (4.4)]

$$\frac{e^{-\pi/2}}{1+e^{-\pi}} + \frac{e^{-2\pi}}{1+e^{-3\pi}} + \frac{e^{-4\pi}}{1+e^{-5\pi}} + \frac{e^{-6\pi}}{1+e^{-7\pi}} \cdots = \sqrt{2\sqrt{2}+4} - (\sqrt{2}+1), \tag{4.2}$$

which is the value of

$$H(q) = \frac{q^{1/2}}{1+q} + \frac{q^2}{1+q^3} + \frac{q^4}{1+q^5} + \frac{q^6}{1+q^7} \cdots$$

at $q = e^{-\pi}$. The proof of (4.2) relies on the analogues of (3.1) and (3.2), namely, [20, (2.16), (2.17)]

$$\sum_{j=0}^{\infty} \frac{(-q; q^2)_j}{(q^2; q^2)_j} q^{j^2} = \frac{1}{(q^4; q^8)_\infty (q; q^8)_\infty (q^7; q^8)_\infty} \tag{4.3}$$

and

$$\sum_{j=0}^{\infty}\frac{(-q;q^2)_j}{(q^2;q^2)_j}q^{j^2+2j}=\frac{1}{(q^4;q^8)_\infty(q^3;q^8)_\infty(q^5;q^8)_\infty}. \tag{4.4}$$

Although there are many proofs of the Rogers-Ramanujan identities (3.1) and (3.2), there are not as many proofs for (4.3) and (4.4). B. Gordon once told the first author that every relation satisfied by the Rogers-Ramanujan continued fraction corresponds to a similar relation satisfied by the Ramanujan-Göllnitz-Gordon continued fraction. This "principle" appears to work for all cases so far. For example, the first two authors of this article have found proofs of (4.3) and (4.4) (see [8]) similar to that of Bressoud's proofs of (3.1) and (3.2). As such, it is natural to wonder if we could prove (4.3) and (4.4) along the same line as proofs of (3.1) and (3.2) given in Section 3. It turns out that Gordon's principle works again. We first state the following analogue of (3.6) for (4.3) and (4.4).

$$\sum_{j=0}^{\infty}\frac{(aq^2/b;q^2)_j}{(q^2;q^2)_j}q^{j^2-j}(-b)^j=\frac{(b;q^2)_\infty}{(a;q^2)_\infty}\sum_{j=0}^{\infty}\frac{(a;q^2)_j(aq^2/b;q^2)_j}{(b;q^2)_j(q^2;q^2)_j}(ab)^j(1-aq^{4j})q^{4j^2-2j}, \tag{4.5}$$

Identity (4.5) is obtained by letting $N\to\infty$ in the following identity.

Theorem 4.1. *Let N be a positive integer, $a\neq q^{-2m}$ and $b\neq q^{-2\ell}$ for non-negative integers m and ℓ. Then*

$$\sum_{j=0}^{N}\frac{(aq^2/b;q^2)_j q^{j^2-j}(-b)^j}{(q^2;q^2)_{N-j}(q^2;q^2)_j}=(b;q^2)_N\sum_{j=0}^{N}\frac{(a;q^2)_j a^j b^j(1-aq^{4j})q^{4j^2-2j}(aq^2/b;q^2)_j}{(q^2;q^2)_j(q^2;q^2)_{N-j}(b;q^2)_j(a;q^2)_{N+j+1}}. \tag{4.6}$$

The proof of (4.6) follows in the same way as the proof of (3.6) and we leave it as an exercise for the reader. Identity (4.5) can be found in [3, p. 84, Entry 4.2.4] while (4.6) appears to be new.

By letting $a\to 1$ and setting $b=-q$, and by setting $a=q^2$ and $b=-q^3$, respectively, in (4.6), we obtain two finite sums

$$\sum_{n=1}^{N}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_{N-n}(q^2;q^2)_n}=(-q;q^2)_N\left(\frac{1}{(q^2;q^2)_N^2}+\sum_{n=0}^{N}\frac{(-1)^n(1+q^{2n})q^{4n^2-n}}{(q^2;q^2)_{N-n}(q^2;q^2)_{N+n}}\right) \tag{4.7}$$

and

$$\sum_{n=0}^{N}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_{N-n}(q^2;q^2)_n}=(-q;q^2)_{N+1}\sum_{n=0}^{N}\frac{(-1)^n(1-q^{2n+1})q^{4n^2+3n}}{(q^2;q^2)_{N-n}(q^2;q^2)_{N+n+1}}. \tag{4.8}$$

Letting $N\to\infty$ and applying the Jacobi triple product, we obtain the two Ramanujan-Göllnitz-Gordon identities (4.3) and (4.4).

5. Ramanujan's cubic continued fraction

A cubic analogue of (4.1) and (4.2) discovered by Ramanujan is given by

$$\frac{e^{-2\pi/3}}{1}\,+\,\frac{e^{-2\pi}+e^{-4\pi}}{1}\,+\,\frac{e^{-4\pi}+e^{-8\pi}}{1}\,+\,\frac{e^{-6\pi}+e^{-12\pi}}{1}\cdots=\frac{-(1+\sqrt{3})+\sqrt{6\sqrt{3}}}{4}, \tag{5.1}$$

which is the special value of Ramanujan's cubic continued fraction

$$G(q)=\frac{q^{1/3}}{1}\,+\,\frac{q+q^2}{1}\,+\,\frac{q^2+q^4}{1}\,+\,\frac{q^3+q^6}{1}\cdots$$

at $q=e^{-2\pi}$. The proof of (5.1), like that of (4.1) and (4.2), requires the following two identities [2, Corollary 6.2.7], which are analogues of (3.1) and (3.2)

$$\frac{(-q;q^2)_\infty(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty}=\sum_{s=0}^{\infty}q^{s^2+2s}\frac{(-q;q^2)_s}{(q^4;q^4)_s} \tag{5.2}$$

and

$$\frac{(-q;q^2)_\infty(q^3;q^3)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty}=\sum_{s=0}^{\infty}q^{s^2}\frac{(-q;q^2)_s}{(q^4;q^4)_s}. \tag{5.3}$$

A natural question, after Section 3 and Section 4, is to ask for proofs of (5.2) and (5.3) similar to the proofs of the pairs of identities ((3.1), (3.2)) and ((4.3), (4.4)). Our attempt to find such proofs for (5.2) and (5.3) leads to the discovery of the following identity which surprisingly allows us to deduce all the three pairs of identities ((3.1), (3.2)), ((4.3), (4.4)) and ((5.2), (5.3)).

Theorem 5.1. *Let N be any nonnegative integer and $a \neq bq^{-m}$ for any positive integer m. Then*

$$\frac{(a;q)_{N+1}}{(aq/b;q)_N} = \sum_{k=0}^{N} \frac{(1-aq^{2k})(q^{-N},a,b;q)_k}{(q,aq/b,aq^{N+1};q)_k}(-a/b)^k q^{k(k+1)/2+kN}. \tag{5.4}$$

The proof of (5.4) is different from the proof of (4.6) and (3.6) and its details are provided as follows:

Proof. We use $f(a)$ to denote the right-hand side of (5.4) and noting that

$$1-aq^{2k} = 1-q^k+q^k(1-aq^k),$$

we find that

$$f(a) = \sum_{k=1}^{N} \frac{(q^{-N},a,b;q)_k}{(q;q)_{k-1}(aq^{N+1},aq/b;q)_k}(-a/b)^k q^{k(k+1)/2+kN} + \sum_{k=0}^{N} \frac{(a;q)_{k+1}(q^{-N},b;q)_k}{(q,aq^{N+1},aq/b;q)_k}(-a/b)^k q^{k(k+3)/2+kN}. \tag{5.5}$$

Making the variable change $k \to k+1$ in the first summation on the right-hand side of (5.5), we find that

$$\begin{aligned} f(a) &= \sum_{k=0}^{N} \frac{(q^{-N},a,b;q)_{k+1}}{(q;q)_k(aq^{N+1},aq/b;q)_{k+1}}(-a/b)^{k+1} q^{(k+1)(k+2)/2+(k+1)N} \\ &\quad + \sum_{k=0}^{N} \frac{(a;q)_{k+1}(q^{-N},b;q)_k}{(q,aq^{N+1},aq/b;q)_k}(-a/b)^k q^{k(k+3)/2+kN} \\ &= \sum_{k=0}^{N} \frac{(a;q)_{k+1}(q^{-N},b;q)_k}{(q,aq^{N+1},aq/b;q)_k}(-a/b)^k q^{k(k+3)/2+kN}\left(1-\frac{aq^{N+1}(1-bq^k)(1-q^{-N+k})}{b(1-aq^{N+k+1})(1-aq^{k+1}/b)}\right). \end{aligned} \tag{5.6}$$

Using

$$\left(1-\frac{aq^{N+1}(1-bq^k)(1-q^{-N+k})}{b(1-aq^{N+k+1})(1-aq^{k+1}/b)}\right) = \frac{(1-aq^{N+1}/b)(1-aq^{2k+1})}{(1-aq^{N+k+1})(1-aq^{k+1}/b)},$$

we rewrite (5.6) as

$$\begin{aligned} f(a) &= (1-aq^{N+1}/b)\sum_{k=0}^{N} \frac{(1-aq^{2k+1})(a;q)_{k+1}(q^{-N},b;q)_k}{(q;q)_k(aq^{N+1},aq/b;q)_{k+1}}(-aq/b)^k q^{k(k+1)/2+kN} \\ &= \frac{(1-a)(1-aq^{N+1}/b)}{(1-aq^{N+1})(1-aq/b)} f(aq). \end{aligned} \tag{5.7}$$

Iterating (5.7), we find that for any non-negative integer n,

$$f(a) = \frac{(a;q)_n(aq^{N+1}/b;q)_n}{(aq^{N+1};q)_n(aq/b;q)_n} f(aq^n).$$

Letting $n \to \infty$ and noting that $f(0)=1$ we find that

$$f(a) = \frac{(a;q)_\infty(aq^{N+1}/b;q)_\infty}{(aq^{N+1};q)_\infty(aq/b;q)_\infty} = \frac{(a;q)_{N+1}}{(aq/b;q)_N}$$

and the proof of (5.4) is complete. □

Remark 5.1. Identity (5.4) can be viewed as a special case of a ${}_6\varphi_5$-summation formula [16, p. 238, (II.21)].

In Section 3, we promise to give a proof of (3.10) without using the q-Chu-Vandermonde identity. We prove (3.10) using (5.4). First, note that (3.10) is a consequence of

$$\frac{1}{(c;q)_N} = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q \frac{c^k q^{k(k-1)}}{(c;q)_k}. \tag{5.8}$$

To prove (5.8), we let $b = aq/c$, followed by letting $a \to 0$ in (5.4).

Identity (5.4) is equivalent to

$$\frac{1}{(q,aq/b;q)_N}=\sum_{n=0}^{N}\frac{(a,b;q)_n}{(q,aq/b;q)_n(q;q)_{N-n}(a;q)_{N+n+1}}(1-aq^{2n})(a/b)^n q^{n^2}.$$

With q replaced by q^2 and b replaced by a/b, we arrive at

$$\frac{1}{(q^2,bq^2;q^2)_N}=\sum_{n=0}^{N}\frac{(a,a/b;q^2)_n(1-aq^{4n})b^n q^{2n^2}}{(q^2,bq^2;q^2)_n(q^2;q^2)_{N-n}(a;q^2)_{N+n+1}}. \tag{5.9}$$

With that, we deduce the following theorem.

Theorem 5.2. *Let N be a positive integer, $b\neq q^{-2m-2}$, $c\neq q^{-2\ell-2}$ and $a\neq q^{-2\nu}$ for non-negative integers m, ℓ and ν. Then*

$$\sum_{n=0}^{N}\frac{(a/c;q^2)_n(-c)^n q^{n^2+n}}{(q^2;q^2)_{N-n}(q^2,bq^2;q^2)_n}=(cq^2;q^2)_N\sum_{n=0}^{N}\frac{(a,a/b,a/c;q^2)_n(1-aq^{4n})(-bc)^n q^{3n^2+n}}{(q^2;q^2)_{N-n}(a;q^2)_{N+n+1}(q^2,bq^2,cq^2;q^2)_n}. \tag{5.10}$$

Proof. The proof is similar to the proof of (3.11), where we used (5.9) instead of (3.8).

Invoking (5.9) on the left-hand side of (5.10), we find that

$$\begin{aligned}
&\sum_{n=0}^{N}\frac{(a/c;q^2)_n(-c)^n q^{n^2+n}}{(q^2;q^2)_{N-n}(q^2,bq^2;q^2)_n}\\
&=\sum_{n=0}^{N}\frac{(a/c;q^2)_n(-c)^n q^{n^2+n}}{(q^2;q^2)_{N-n}}\sum_{k=0}^{n}\frac{(a,a/b;q^2)_k(1-aq^{4k})b^k q^{2k^2}}{(q^2,bq^2;q^2)_k(q^2;q^2)_{n-k}(a;q^2)_{n+k+1}}\\
&=\sum_{k=0}^{N}\frac{(a,a/b;q^2)_k(1-aq^{4k})b^k q^{2k^2}}{(q^2,bq^2;q^2)_k}\sum_{n=k}^{N}\frac{(a/c;q^2)_n(-c)^n q^{n^2+n}}{(q^2;q^2)_{N-n}(q^2;q^2)_{n-k}(a;q^2)_{n+k+1}}\\
&=\sum_{k=0}^{N}\frac{(a,a/b,a/c;q^2)_k(1-aq^{4k})(-bc)^k q^{3k^2+k}}{(q^2,bq^2;q^2)_k(a;q^2)_{2k+1}(q^2;q^2)_{N-k}}\sum_{m=0}^{N-k}\frac{(aq^{2k}/c;q^2)_m(-cq^{2k+2})^m q^{m^2-m}}{(aq^{4k+2};q^2)_m}\begin{bmatrix}N-k\\ m\end{bmatrix}_{q^2}\\
&=\sum_{k=0}^{N}\frac{(a,a/b,a/c;q^2)_k(1-aq^{4k})(-bc)^k q^{3k^2+k}}{(q^2,bq^2;q^2)_k(a;q^2)_{2k+1}(q^2;q^2)_{N-k}}\frac{(cq^{2k+2};q^2)_{N-k}}{(aq^{4k+2};q^2)_{N-k}},
\end{aligned} \tag{5.11}$$

where, in the last step, we used the q-analogue of the Chu-Vandermonde identity (3.5) with q replaced by q^2, a replaced by aq^{2k}/c, $c=aq^{4k+2}$ and $n=N-k$.

Simplifying (5.11) further then yields the right-hand side of (5.10) and this completes the proof. $\square$

If we let $N\to\infty$ in (5.10), we deduce that

$$\sum_{n=0}^{\infty}\frac{(a/c;q^2)_n(-c)^n q^{n^2+n}}{(q^2,bq^2;q^2)_n}=\frac{(cq^2;q^2)_\infty}{(a;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a,a/b,a/c;q^2)_n(1-aq^{4n})(-bc)^n q^{3n^2+n}}{(q^2,bq^2,cq^2;q^2)_n}. \tag{5.12}$$

Identity (5.12) is equivalent to an identity found in Ramanujan's Lost Notebook (see [3, Entry 4.2.3]. Identity (5.10), however, appears to be new.

From (5.10), we immediately deduce the following corollary.

Corollary 2. *For $|q|<1$ and positive integer N, we have*

$$\sum_{n=0}^{N}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_{N-n}(q^4;q^4)_n}=(-q;q^2)_N\left(\frac{1}{(q^2;q^2)_N^2}+\sum_{n=1}^{N}\frac{2(-1)^n q^{3n^2}}{(q^2;q^2)_{N-n}(q^2;q^2)_{N+n}}\right) \tag{5.13}$$

and

$$\sum_{n=0}^{N}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_{N-n}(q^4;q^4)_n}=(-q;q^2)_{N+1}\sum_{n=0}^{N}\frac{(1-q^{2n+1})(-1)^n q^{3n^2+2n}}{(q^2;q^2)_{N-n}(q^2;q^2)_{N+n+1}}. \tag{5.14}$$

Proof. By letting $a\to 1$ and setting $b=-1$, $c=-1/q$ in (5.10) we obtain (5.13). Identity (5.14) is obtained by setting $a=q^2$, $b=-1$, and $c=-q$ in (5.10). $\square$

Letting $N\to\infty$, dividing both sides by $(q^2;q^2)_\infty$ and applying the Jacobi triple product identity in each of (5.13) and (5.14), we obtain (5.3) and (5.2), respectively.

We next observe that by letting $b\to 0$ and $c=b/q^2$, we obtain another proof of (4.6). We have seen that (4.6) implies (4.3) and (4.4).

If we let $b\to 0$ and replace q^2 by q in (4.6), we arrive at (3.11). We have already seen that (3.11) gives (3.6) as $N\to\infty$. As mentioned in Section 3, (3.6) implies (3.1) and (3.2). We have thus shown that (5.10) implies all the identities associated with the Rogers-Ramanujan continued fraction, the Ramanujan-Göllnitz-Gordon continued fraction and Ramanujan's cubic continued fraction.

In Section 3, we deduce a finite version of the quintuple product identity (3.14) from (3.6). The same approach illustrated there can be applied using (5.10) instead of (3.14) to deduce a generalization of (3.6).

We replace N by $2N$ and the summation index n by $n+N$ in (5.10), and multiply both sides by $(q^2;q^2)_{2N}$. After simplifying, we arrive at

$$\begin{aligned}&\sum_{n=-N}^{N}\begin{bmatrix}2N\\N+n\end{bmatrix}_{q^2}\frac{(a/c;q^2)_{N+n}(-c)^nq^{2Nn+n^2+n}}{(bq^2;q^2)_{N+n}}=(cq^2;q^2)_{2N}\\&\times\sum_{n=-N}^{N}\begin{bmatrix}2N\\N+n\end{bmatrix}_{q^2}\frac{(a,a/b,a/c;q^2)_{N+n}(1-aq^{4N+4n})b^{N+n}(-c)^nq^{2N^2+6Nn+3n^2+n}}{(a;q^2)_{3N+n+1}(bq^2,cq^2;q^2)_{N+n}}.\end{aligned}$$

Setting $a=xq^{-4N}$, $b=yq^{-2N}$, and $c=zq^{-2N}$, we find that

$$\begin{aligned}&\sum_{n=-N}^{N}\begin{bmatrix}2N\\N+n\end{bmatrix}_{q^2}\frac{(xq^{-2N}/z;q^2)_{N+n}(-z)^nq^{n^2+n}}{(yq^{-2N+2};q^2)_{N+n}}=(zq^{-2N+2};q^2)_{2N}\\&\times\sum_{n=-N}^{N}\begin{bmatrix}2N\\N+n\end{bmatrix}_{q^2}\frac{(xq^{-2N}/y,xq^{-2N}/z;q^2)_{N+n}(1-xq^{4n})y^{N+n}(-z)^nq^{2Nn+3n^2+n}}{(xq^{-2N+2n};q^2)_{2N+1}(yq^{-2N+2},zq^{-2N+2};q^2)_{N+n}}.\end{aligned}\tag{5.15}$$

Noting that for any positive integer N and any integer n,

$$\begin{aligned}(aq^{-2N};q^2)_{N+n}&=(aq^{-2N};q^2)_N(a;q^2)_n\\&=(-1)^Na^Nq^{-N^2-N}(q^2/a;q^2)_N(a;q^2)_n,\\(zq^{-2N+2};q^2)_{2N}&=(-1)^Nz^Nq^{-N^2+N}(1/z;q^2)_N(zq^2;q^2)_N\end{aligned}$$

and

$$\begin{aligned}(xq^{-2N+2n};q^2)_{2N+1}&=(xq^{-2N+2n};q^2)_{N-n}(x;q^2)_{N+n+1}\\&=(-1)^{N-n}x^{n-N}q^{-N^2+2Nn-N-n^2+n}(q^2/x;q^2)_{N-n}(x;q^2)_{N+n+1},\end{aligned}$$

we see that (5.15) is equivalent to

$$\begin{aligned}&\sum_{n=-N}^{N}\begin{bmatrix}2N\\N+n\end{bmatrix}_{q^2}\frac{(q^2z/x;q^2)_N(x/z;q^2)_n(x/(yz))^N(-z)^nq^{n^2+n-2N}}{(1/y;q^2)_N(yq^2;q^2)_n}\\&=(1/z,zq^{\cdot}q^2)_N\sum_{n=-N}^{N}\begin{bmatrix}2N\\N+n\end{bmatrix}_{q^2}\frac{(yq^2/x,zq^2/x;q^2)_N(x/z,x/y;q^2)_n(1-xq^{4n})(-1)^nx^n(x/(yz))^Ny^n(-z)^nq^{-2N+4n^2}}{(q^2/x;q^2)_{N-n}(x;q^2)_{N+n+1}(1/y,1/z;q^2)_N(yq^2,zq^2;q^2)_n}.\end{aligned}$$

Upon further simplification and rearrangement, we arrive at

$$\begin{aligned}&\sum_{n=-N}^{N}\begin{bmatrix}2N\\N+n\end{bmatrix}_{q^2}\frac{(x/z;q^2)_n(-z)^nq^{n^2+n}}{(yq^2;q^2)_n}\\&=(yq^2/x,zq^2;q^2)_N\sum_{n=-N}^{N}\begin{bmatrix}2N\\N+n\end{bmatrix}_{q^2}\frac{(x/z,x/y;q^2)_n(1-xq^{4n})x^ny^nz^nq^{4n^2}}{(q^2/x;q^2)_{N-n}(x;q^2)_{N+n+1}(yq^2,zq^2;q^2)_n}.\end{aligned}\tag{5.16}$$

Letting $y,z\to 0$ and replacing q^2 by q, we recover (3.14). This shows that (5.16) is a generalization of (3.14).

If we let $N \to \infty$, replace q^2 by q and rearrange, we obtain an interesting identity given by

$$\frac{(q/x, x; q)_\infty}{(yq/x, zq; q)_\infty} \sum_{j=-\infty}^{\infty} \frac{(x/z; q)_j (-z)^j q^{(j^2+j)/2}}{(yq; q)_j} = \sum_{j=-\infty}^{\infty} \frac{(x/z, x/y; q)_j (1 - xq^{2j})(xyz)^j q^{2j^2}}{(yq, zq; q)_j}. \tag{5.17}$$

Letting $y \to 0$ and $z \to 0$ in (5.17) and using the Jacobi triple product identity (3.13), we obtain the quintuple product identity (3.22). This shows that (5.17) is a generalization of the quintuple product identity.

Next, we observe that the right-hand side of (5.17) is symmetric in y and z and we immediately deduce (1.1). This shows that (1.1) is a generalization of (1.2) and the proof of (1.2) given here is new. When we set $z \to 1$, we arrive at

$$\sum_{j=-\infty}^{\infty} \frac{(-1)^j q^{(j^2+j)/2}}{1 - yq^j} = \frac{(q, q; q)_\infty}{(q/y, y; q)_\infty}. \tag{5.18}$$

6. On the q-Chu-Vandermonde identity and its consequences

The identity (3.5) is used several times in this article. In this section, we give a short proof of (3.5) which is independent of the q-binomial theorem. This proof is motivated by the proof of (5.4).

Proof of (3.5)**.** It is obvious that when $N = 0$, both sides of (3.5) equal 1. Now we assume $N \ge 1$. Recall that for $N \ge 1$,

$$\begin{bmatrix} N \\ n \end{bmatrix}_q = q^n \begin{bmatrix} N-1 \\ n \end{bmatrix}_q + \begin{bmatrix} N-1 \\ n-1 \end{bmatrix}_q. \tag{6.1}$$

For brevity we use $F_N(c)$ to denote the left-hand side of (3.5) and substituting (6.1) into $F_N(c)$ we deduce that

$$F_N(c) = \sum_{n=0}^{N} \begin{bmatrix} N-1 \\ n \end{bmatrix}_q (-1)^n \frac{(a;q)_n}{(c;q)_n} \left(\frac{c}{a}\right)^n q^{n(n+1)/2} + \sum_{n=0}^{N} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix}_q (-1)^n \frac{(a;q)_n}{(c;q)_n} \left(\frac{c}{a}\right)^n q^{n(n-1)/2}.$$

Noting that

$$\begin{bmatrix} N-1 \\ N \end{bmatrix}_q = \begin{bmatrix} N-1 \\ -1 \end{bmatrix}_q = 0,$$

we have

$$F_N(c) = \sum_{n=0}^{N-1} \begin{bmatrix} N-1 \\ n \end{bmatrix}_q (-1)^n \frac{(a;q)_n}{(c;q)_n} \left(\frac{c}{a}\right)^n q^{n(n+1)/2} + \sum_{n=1}^{N} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix}_q (-1)^n \frac{(a;q)_n}{(c;q)_n} \left(\frac{c}{a}\right)^n q^{n(n-1)/2}. \tag{6.2}$$

Replacing $n-1$ by n in the second series of the right-hand side of (6.2), we conclude that

$$\begin{aligned} F_N(c) &= \sum_{n=0}^{N-1} \begin{bmatrix} N-1 \\ n \end{bmatrix}_q (-1)^n \frac{(a;q)_n}{(c;q)_n} \left(\frac{c}{a}\right)^n q^{n(n+1)/2} - \sum_{n=0}^{N-1} \begin{bmatrix} N-1 \\ n \end{bmatrix}_q (-1)^n \frac{(a;q)_{n+1}}{(c;q)_{n+1}} \left(\frac{c}{a}\right)^{n+1} q^{n(n+1)/2} \\ &= \sum_{n=0}^{N-1} \begin{bmatrix} N-1 \\ n \end{bmatrix}_q (-1)^n \frac{(a;q)_n}{(c;q)_n} \left(\frac{c}{a}\right)^n q^{n(n+1)/2} \left(1 - \frac{c(1-aq^n)}{a(1-cq^n)}\right) \\ &= \frac{(1-c/a)}{(1-c)} \sum_{n=0}^{N-1} \begin{bmatrix} N-1 \\ n \end{bmatrix}_q (-1)^n \frac{(a;q)_n}{(qc;q)_n} \left(\frac{qc}{a}\right)^n q^{n(n-1)/2}. \end{aligned}$$

It follows that

$$F_N(c) = \frac{(1-c/a)}{(1-c)} F_{N-1}(qc).$$

Noting that $F_0(c) = 1$ for any c, iterating the above equation and using $F_0(q^N c) = 1$ we find that

$$F_N(c) = \frac{(c/a;q)_N}{(c;q)_N} F_0(q^N c) = \frac{(c/a;q)_N}{(c;q)_N}. \quad \square$$

We next show the relation between (3.5) and the Jacobi triple product identity.

Replacing N by $2N$ and making the change of variable $m \to N+m$ in (3.5), we find that

$$\frac{(c/b;q)_{2N}}{(c;q)_{2N}} = \sum_{m=-N}^{N} \begin{bmatrix} 2N \\ N+m \end{bmatrix}_q (-1)^{m+N} q^{m(m-1)/2+mN+N(N-1)/2} \frac{(b;q)_{N+m}}{(c;q)_{N+m}} \left(\frac{c}{b}\right)^{N+m},$$

Next, setting $c = abq^{-N}$, we arrive at

$$\frac{(aq^{-N};q)_{2N}}{(abq^{-N};q)_{2N}} = \sum_{m=-N}^{N} \begin{bmatrix} 2N \\ N+m \end{bmatrix}_q (-1)^{m+N} q^{m(m-1)/2+mN+N(N-1)/2} \frac{(b;q)_{N+m}}{(abq^{-N};q)_{N+m}} \left(aq^{-N}\right)^{N+m}.$$

By applying

$$(xq^{-N};q)_{N+k} = (-1)^N q^{-N(N+1)/2} x^N (q/x;q)_N (x)_k,$$

and simplifying, we arrive at

$$\frac{(a,q/a;q)_N}{(ab;q)_N} = \sum_{m=-N}^{N} \begin{bmatrix} 2N \\ N+m \end{bmatrix}_q (-1)^m a^m q^{m(m-1)/2} \frac{(b;q)_{N+m}}{(ab;q)_m}. \tag{6.3}$$

Letting $N \to \infty$ in (6.3), we arrive at

$$\frac{(a,q/a,q;q)_\infty}{(ab,b;q)_\infty} = \sum_{m=-\infty}^{\infty} \frac{(-1)^m a^m q^{m(m-1)/2}}{(ab;q)_m}. \tag{6.4}$$

If we let $b = q/a$ in (6.4), we arrive at Euler's identity

$$(a;q)_\infty = \sum_{j=0}^{\infty} \frac{(-1)^j q^{(j^2-j)/2} a^j}{(q;q)_j}$$

since $1/(q;q)_j = 0$ for $j < 0$.

If we let $b \to 0$, then we recover the Jacobi triple product identity

$$(a,q/a,q;q)_\infty = \sum_{m=-\infty}^{\infty} (-1)^m a^m q^{m(m-1)/2}.$$

In other words, we note that (6.4) generalizes both Euler's identity and Jacobi's triple product identity.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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